Naval Research Laboratory

Washington, DC 20375-5000 NRL Report 8995 August 21, 1986

Analytical and Physical Aspects of Two-Dimensional Spectra Associated with Stationary Random Processes

JOHN M. BERGIN

Acoustics Media Characterization Branch
Acoustics Division

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Report 8995	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Research Laboratory 6b. OFFICE SYMBOL (If applicable) Code 5100	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) Washington, DC 20375-5000	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Office of Naval Research 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code) Arlington, VA 22314	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61153N		RR032-01-41	DN 180-028

11. TITLE (Include Security Classification)
Analytical and Physical Aspects of Two-Dimensional Spectra Associated with Stationary Random Processes

12. PERSONAL AUTHOR(S)
Bergin, John M.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Interim	FROM ______ TO ______	1986 August 25	27

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Spectral analysis
			Random processes
			Wavenumber

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The use of spectral analysis techniques for the study of stochastic processes that occur over multidimensional spaces, such as ocean wind-waves or ocean bottom roughness, leads to a large number of possible spectral quantities one can use in a description of the process. The multidimensional case presents difficulties since this variety does not arise in the study of processes that depend on one dimension. We discuss the specific case of a process taking place over two dimensions that applies to the nature of the ocean surface at a particular instant in time or to the nature of the ocean bottom. The several possible spectral quantities of interest are described and related to the basic two-dimensional wavenumber spectrum. A physical interpretation of the spectral quantities is developed by relating them to the distribution of spectral energy in the two-dimensional wavenumber plane. Some specific analytical results are obtained for the important special case of isotropy, particularly with respect to power-law models for the two-dimensional spectrum. By use of the ocean surface wind-wave case as an example, the investigation of anisotropy for stochastic processes must be considered as an issue of equal importance to stationarity of the process.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL John M. Bergin	22b. TELEPHONE (Include Area Code) (202) 767-2024 22c. OFFICE SYMBOL Code 5100

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE

☆U.S. Government Printing Office: 1985-507-047

CONTENTS

PREFACE

This work is related to an ocean-bottom research program at NRL. The program is motivated by the recent development of a variety of bathymetric measurement systems that resolve aspects of bottom variability previously beyond the realm of quantitative study. The general aim of the research is to develop statistical/stochastic representations of ocean-bottom variability at scales that are relevant to underwater acoustic systems. Such representations will allow the prediction of acoustic system performance and lead to the better design and use of underwater acoustic systems that either utilize or are sensitive to acoustic interaction with the ocean bottom.

ANALYTICAL AND PHYSICAL ASPECTS OF TWO-DIMENSIONAL SPECTRA ASSOCIATED WITH STATIONARY RANDOM PROCESSES

INTRODUCTION

This report is an exploration of certain fundamental and general aspects of spectral relationships that arise in the analysis of random, or stochastic, processes that depend on more than one dimension. The multidimensional spectrum is the central focus of our remarks rather than the correlation function. A main point of the discussion is that several types of spectra may arise in the analysis of the multidimensional situation, which can be a source of confusion. There is a significant difference with the one-dimensional situation which is the case most often treated in the literature. Stochastic processes that occur over more than one dimension offer a much greater richness in terms of statistical quantities that can be employed for their description. We stress general aspects that may enter into the considerations of many specific applications. Furthermore, the physical interpretation of the various quantities that occur is stressed as opposed to a more formal mathematical presentation.

There is a wide prevalence of power-law representations used to represent spectra in virtually every field in which spectra are used. This is undoubtedly due to the ease of analytically dealing with such simple forms and they further offer a concise means of summarizing spectral behavior. We evaluate a function that relates various spectral forms in the case of power-law behavior and investigate its analytical behavior.

Although much of the discussion is aimed at general properties, we give some presentation of the ocean surface wave case and suggest that it can be viewed as a paradigm for the multidimensional case. This suggestion is partly motivated by the long historical evolution of attempts to create a description of the dynamics of the sea surface, which has led to a rich literature devoted to probabilistic, or stochastic, models of the sea surface [1,2]. Reference 3 discusses the different viewpoints and problems associated with the early struggle to fashion a statistical approach to ocean wave spectra. The main point of the suggestion is that knowledge of the ocean surface wave literature may be a useful aid in the description and investigation of other processes.

VARIOUS FORMS OF SPECTRAL ENERGY AND THEIR PHYSICAL INTERPRETATION

Consider a stochastic process that depends on two independent variables which we designate x and y. The primary question of interest is the description of the spectral structure associated with $\eta\,(x,y)$, where η represents some physical aspect of the process. Thus one can say that η evolves in a random fashion as a function of x and y. We further assume that the process is stationary [4] so that the statistical structure does not vary with location in the plane. This assumption enables one to speak in a meaningful way about statistical averages associated with the process. Averages are denoted by an overbar. As a matter of convenience, we assume that

$$\overline{\eta} = 0,$$

so that we are dealing with fluctuations about a zero level.

Manuscript approved April 17, 1986.

It is possible to represent the process in terms of a Fourier-Stieltjes integral

$$\eta(\mathbf{x}) = \int_{\mathbf{k}} dA(\mathbf{k}) e^{i\mathbf{kx}}, \tag{1}$$

where the integration is over all wavenumber space. Certain consequences of this representation are our main concern. The Fourier-Stieltjes representation is easily interpreted as a superposition of an infinite number of infinitesimal, sinusoidal waves of different wavenumbers with different directions. The component associated with wavenumber $\mathbf{k}$ has an amplitude $dA(\mathbf{k})$ and an orientation in physical space specified by $\mathbf{k}$. It is convenient to describe some aspects of the representation in terms of wavenumber space and other aspects in terms of physical space as indicated in Fig. 1.

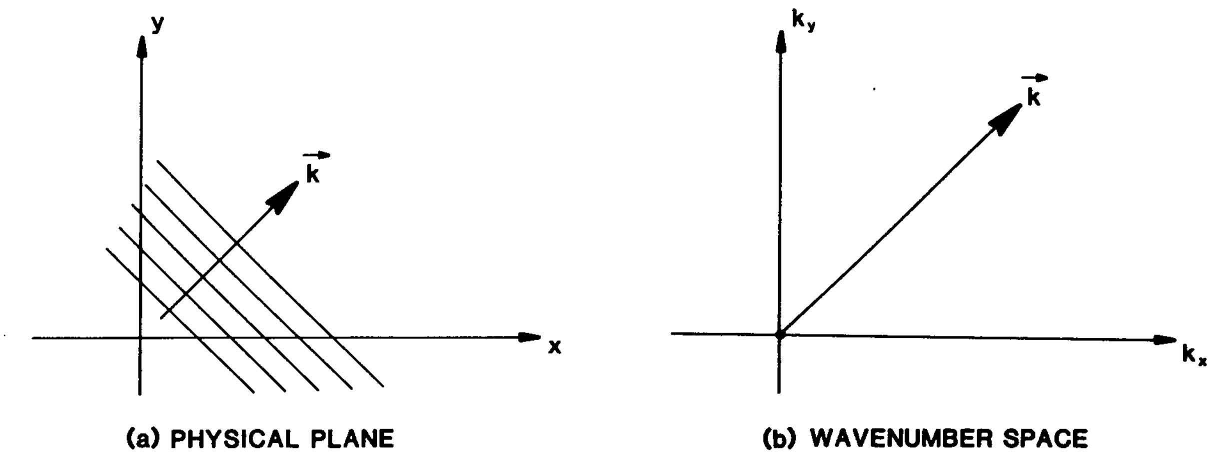

Fig. 1 — Relationship between physical space and wavenumber space for the Fourier-Stieltjes representation of the stochastic process. In (a) the crests associated with the wave component $\vec{k}$ are indicated. The crests are perpendicular to the direction $\vec{k}$ and have a spacing $1/|\vec{k}|$. The wave component shown arises from a point in wavenumber space.

A fundamental property of the Fourier-Stieltjes coefficients is that

$$\begin{aligned} \overline{dA(\mathbf{k})dA^*(\mathbf{k}')} &= 0 \text{ if } \mathbf{k} \neq \mathbf{k}' \\ &= S(\mathbf{k})d\mathbf{k} \text{ if } \mathbf{k} = \mathbf{k}'. \end{aligned} \tag{2}$$

From the representation in Eq. (1) it can be shown that

$$\overline{\eta^2} = \int_{\mathbf{k}} |dA(\mathbf{k})|^2,$$

or using Eq. (2),

$$\overline{\eta^2} = \int_{\mathbf{k}} S(\mathbf{k})d\mathbf{k}. \tag{3}$$

Thus $S(\mathbf{k})$ gives the density of contributions to the mean square per unit area of wavenumber space. $S(\mathbf{k})$ is referred to as the wavenumber spectrum, or power spectral density, of the process. The wavenumber spectrum tells us how the mean square value is distributed over the wavenumber plane.

Some additional properties of the coefficients should be noted. Equation (2) indicates that different components are uncorrelated as a consequence of the stationarity assumption. They are not necessarily independent. We further assume that η measures some real valued aspect of the process, and this implies that

$$dA^*(\mathbf{k}) = dA(-\mathbf{k});$$

however, since

$$S(\mathbf{k})d\mathbf{k} = |dA(\mathbf{k})|^2,$$

we have

$$S(-\mathbf{k})\,d\mathbf{k} = |dA(-\mathbf{k})|^2 = |dA^*(\mathbf{k})|^2 = |dA(\mathbf{k})|^2 = S(\mathbf{k})\,d\mathbf{k},$$

so that

$$S(-\mathbf{k}) = S(\mathbf{k}). \tag{3}$$

One would initially think that the Fourier-Stieltjes components should have no constraints as far as their behavior in the wavenumber plane is concerned. This is true because such constraints produce restrictions on the nature of the directionality of the components. However, in the present case, the process η $(\mathbf{x})$ consists of an ensemble of functions each of which represents a surface that is not changing in time. Consequently, while the surface may be represented in terms of component sinusoidal variations, the sinusoidal components are not propagating, and it is therefore impossible to distinguish a sinusoid associated with $+\mathbf{k}$ from one associated with $-\mathbf{k}$. The Fourier-Stieltjes representation simply assigns an equal amplitude (and energy) at both wavenumber $\mathbf{k}$ and $-\mathbf{k}$ to produce a real valued sinusoidal oscillation of the form

$$\sin\,(k_x x + k_y y + \phi).$$

Later we discuss the specific case of ocean surface wind-waves where it is possible to invoke a certain assumption about the nature of the process to remove the apparent ambiguity.

This was a summary of the important properties of the Fourier-Stieltjes representation of the process. A mathematical justification with varying levels of rigor can be found in several references [5-7]. The important quantity for the following discussion is the spectrum $S(\mathbf{k})$, which can also be written in terms of the wavenumber components as

$$S(\mathbf{k}) = S(k_x,\, k_y),$$

and the relation of this spectrum to the mean square $\overline{\eta^2}$ cited in Eq. (3). We investigate several quantities that are associated with $S(\mathbf{k})$. An important ingredient is the physical interpretation of the quantities and how they are related to the fundamental spectrum $S(\mathbf{k})$.

It is sometimes convenient to describe the distribution of spectral energy in terms of polar coordinates in the spectral plane. This form would be useful if one wanted to investigate the distribution of spectral energy as a function of angle in the wavenumber plane (and would correspond to directional properties of the process in physical space). The desired spectrum would have the form $S(k,\,\theta)$, where k is the magnitude of the wavenumber and θ is an angle in the spectral plane measured counterclockwise from the k_x axis. For this spectrum we have

$$\overline{\eta^2} = \int_0^\infty \int_0^{2\pi} S(k,\,\theta)\,d\theta dk. \tag{4}$$

We can refer to $S(k,\,\theta)$ as a directional spectrum. It can be thought of as a distribution of mean energy in k, θ space as indicated in Fig. 2 where a differential area centered on k, θ contributes an amount $S(k,\,\theta)\,d\theta dk$ to the mean square.

To deduce $S(k,\,\theta)$ from $S(k_x,\,k_y)$ it is useful to think of a small differential angular sector in the k_x, k_y plane (see Fig. 2(a)). Its contribution to the mean square is $S(k,\,\theta)\,d\theta dk$, which can also be written as

$$S(k,\,\theta)\,d\theta dk = \iint_{sector} S(k_x,\,k_y)\,dk_x dk_y.$$

When the differential sector is sufficiently small $S(k_x,\,k_y)$ will be effectively constant over the differential sector so that

$$S(k,\,\theta)\,d\theta dk = S(k_x,\,k_y) \iint_{sector} dk_x dk_y = S(k_x,\,k_y)\,k d\theta dk.$$

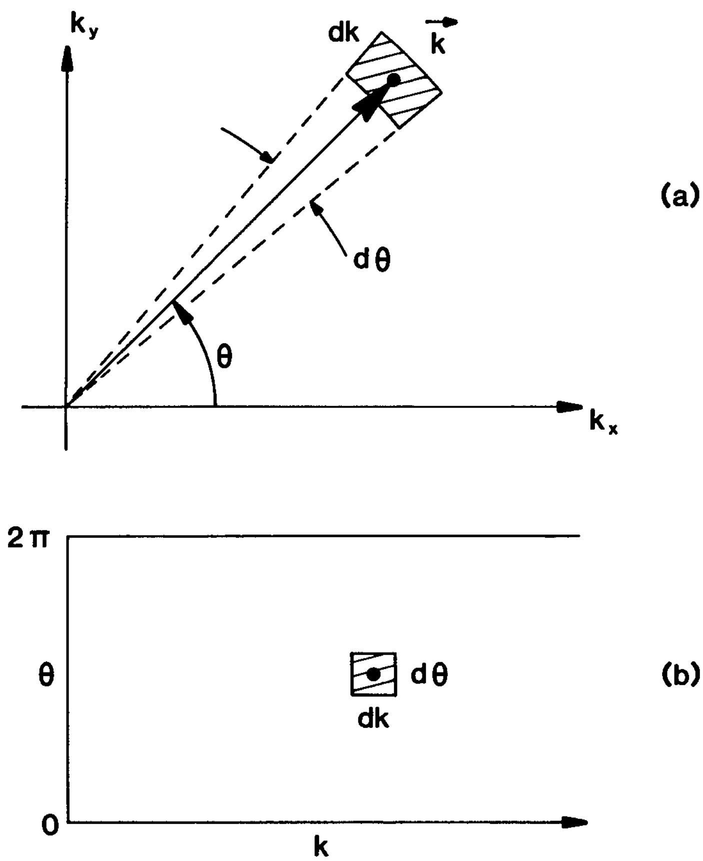

Fig. 2 — The plane associated with $S(k_x, k_y)$ in (a) and that associated with $S(k, \theta)$ in (b). $S(k, \theta)$ arises from a transformation of the k_x, k_y space to k, θ space. The contribution to the mean square from the differential area in (a) is equal to the contribution from the differential area in (b).

This yields the relation

$$S(k, \theta) = kS(k, \theta) \tag{5}$$

in which $S(k, \theta)$ designates the function $S(k_x, k_y)$ expressed in terms of the polar coordinates. The relation between these two spectra is thus established by considering the transformation from one space to another or, equivalently, from one representation to another.

Another spectrum of interest arises when we describe the contribution to the mean square made by wavenumber components that are restricted to bands to wavenumber magnitude. The situation is indicated in Fig. 3. We can define the contribution to $\overline{\eta^2}$ from a differential band of wavenumbers as $S(k)dk$ so that

$$\overline{\eta^2} = \int_0^\infty S(k)\,dk. \tag{6}$$

The spectrum $S(k)$ is sometimes referred to as a scalar wavenumber spectrum, which is suggestive of its origin as a spectral quantity that describes spectral energy irrespective of orientation of the wavenumber. It is often used in the discussion of processes which depend on more than one spatial dimension. It is particularly useful in those situations where the basic process is isotropic with the fundamental spectrum $S(k_x, k_y)$ being a function only of the magnitude of the wavenumber. For the general situation we have that

$$S(k)\,dk = \iint_{band} S(k_x, k_y)\,dk_x\,dk_y. \tag{7}$$

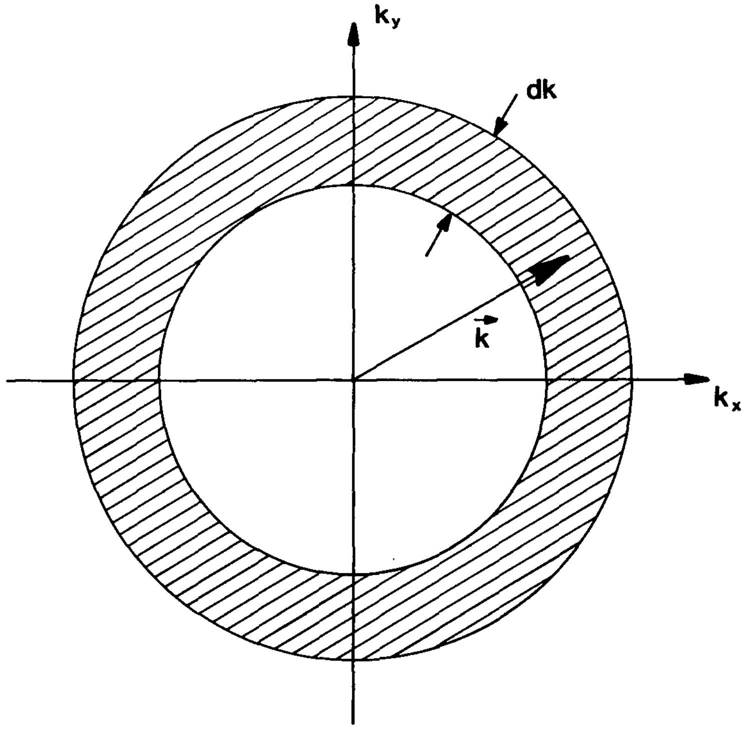

Fig. 3 — Sketch showing the differential band in wavenumber space which contributes an amount $S(k)dk$ to the mean square $\overline{\eta^2}$

Converting the integration in Eq. (7) to polar coordinates yields

$$S(k)dk = \int_0^{2\pi} S(k, \theta)kd\theta dk,$$

so that

$$S(k) = \int_0^{2\pi} S(k, \theta)kd\theta. \tag{8}$$

This gives the relation of $S(k)$ to the fundamental two-dimensional spectrum of the process.

The scalar wavenumber spectrum can be thought of as a one-dimensional spectrum in the sense that it describes the distribution of spectral energy over the one-dimensional axis of k. Part of its usefulness stems from the reduction to one-dimension of the more complicated spectral energy distribution in wavenumber space. However, note that it refers to a process defined over two dimensions.

It is also possible to associate one-dimensional spectra with the process by asking for the contribution to the mean square made by spectral components having a k_x component in a small band of k_x values. We can write

$$S(k_x)dk_x = \left[\int_{-\infty}^{\infty} S(k_x, k_y)dk_x \right] dk_y$$

so that the one-dimensional spectral density is

$$S(k_x) = \int_{-\infty}^{\infty} S(k_x, k_y)dk_y. \tag{9}$$

Figure 4 shows the region of wavenumber space that contributes to the energy $S(k_x)dk_x$. The mean-square is given by

$$\overline{\eta^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k_x, k_y)dk_x dk_y,$$

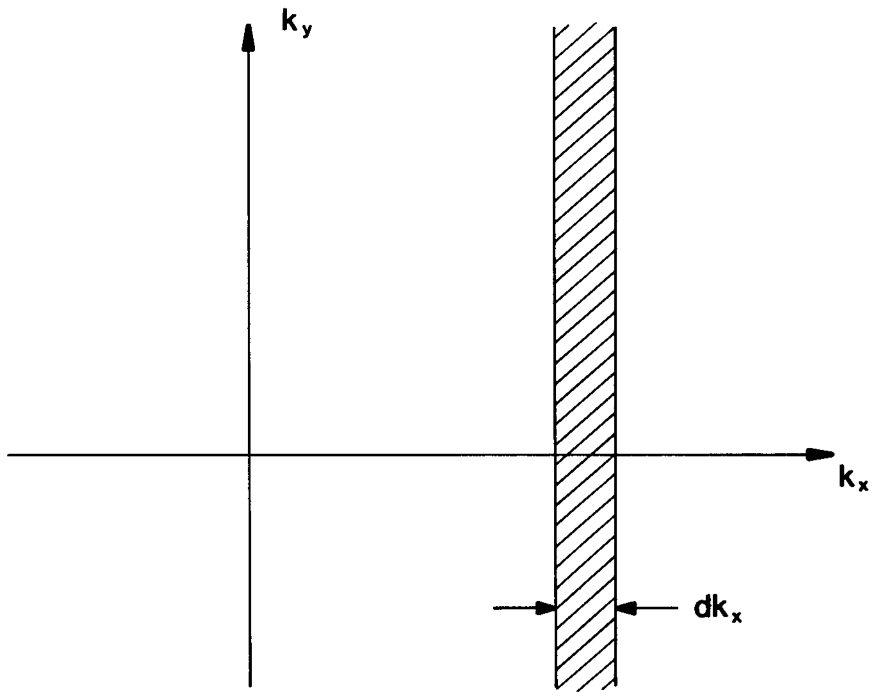

Fig. 4 — Schematic indication of the region of wavenumbers which make a contribution to $S(k_x)dk_x$, where $S(k_x)$ is the one-dimensional spectrum associated with the x-direction

which can also be written as

$$\overline{\eta^2} = \int_{-\infty}^{\infty} S(k_x)\,dk_x \tag{10}$$

in terms of the one-dimensional spectrum $S(k_x)$.

The symmetry property of $S(k_x, k_y)$ leads to a symmetry property of the one-dimensional spectrum. This can be derived by evaluating

$$S(-k_x) = \int_{-\infty}^{\infty} S(-k_x, k_y)\,dk_y.$$

From Eq. (3)

$$S(-k_x, k_y) = S(k_x, -k_y)\,,$$

so that

$$S(-k_x) = \int_{-\infty}^{\infty} S(k_x, -k_y)\,dk_y.$$

The integration can be carried out by changing variables to $k_y' = -k_y$ to get

$$S(-k_x) = \int_{-\infty}^{\infty} S(k_x, k_y')\,dk_y'$$

and, when compared to Eq. (9), we conclude

$$S(-k_x) = S(k_x). \tag{11}$$

This establishes the symmetry property for the one-dimensional spectrum.

An important consequence of Eq. (9) to remember is that the one-dimensional spectrum $S(k_x)$ contains contributions from an infinite range of k_y. This is easily pictured in terms of Fig. 5, which shows a track parallel to the x-axis along which measurements of the process are made and from which the one-dimensional spectrum can be calculated. It is clear from Fig. 5 that all spectral components of the process contribute to the variability of $\eta(x, y)$ along the sampling track. Reference 7 describes the

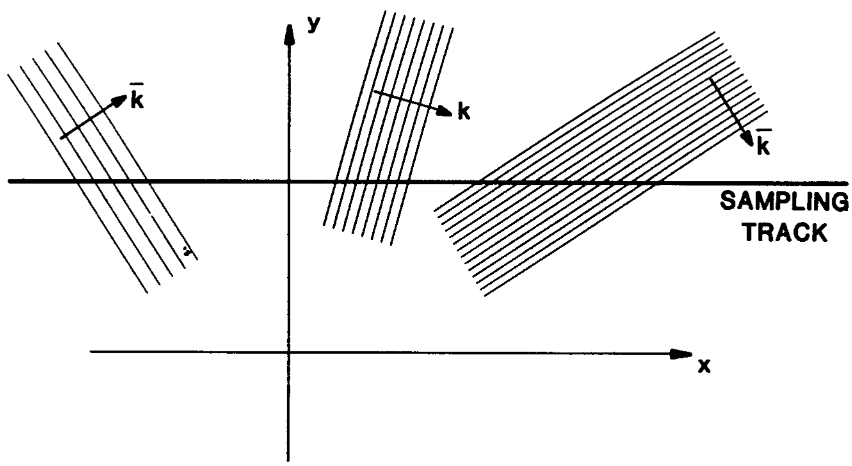

Fig. 5 — Schematic illustration to demonstrate how all spectral components of the two-dimensional process contribute to the variability along a track parallel to the x-axis. Each of the spectral components is defined throughout the plane and are only partially indicated in the figure.

situation by terming the one-dimensional spectrum as an "aliased" version of the two-dimensional spectrum. Although we have restricted attention to the one-dimensional spectrum $S(k_x)$, it is clear that an infinite number of one-dimensional spectra can be derived from the process by measuring the process along lines at various orientations in the physical plane. Each of the possible one-dimensional spectra would have the same dependence on all components of the process as indicated for $S(k_x)$.

In this section we have considered a variety of spectra that can be associated with the random process $\eta(x, y)$ and have attempted to stress the physical interpretation of the spectra as well as their relation to the fundamental wavenumber spectrum $S(k_x, k_y)$. As the complexity of the process increases, the number of possible spectra increase and can pose a serious problem to understanding. The problem is compounded by the lack of a standardized terminology. It is essential to keep in mind a clear interpretation of the particular spectral quantity and its relation to the complete process being described.

RELATIONSHIP OF THE ONE-DIMENSIONAL SPECTRUM TO VARIOUS TWO-DIMENSIONAL SPECTRA

It is shown in the previous section that if the basic two-dimensional spectrum is known, a variety of other spectra may be related to it. A significant question to consider is what one can infer about the two-dimensional spectrum from knowledge of the one-dimensional spectrum. For this question there appears to be no answer that can be stated in general terms other than to say that knowledge of the one-dimensional spectrum places constraints on the possible form of the two-dimensional spectrum. We shall consider a special, but important, case of the question to which a definite answer may be given.

We first start with an expression giving the one-dimensional spectrum in terms of the two-dimensional spectrum expressed in terms of polar coordinates. We can write Eq. (9) as

$$S(k_x) = \int_{-\infty}^{\infty} S(k,\theta)\, dk_y. \tag{12}$$

It is convenient to change the integration from one k_y to polar coordinates using the substitutions

$$k_y = \left(k^2 - k_x^2\right)^{1/2}, \tag{13}$$

when $k_y \geqslant 0$ and

$$k_y = -\left(k^2 - k_x^2\right)^{1/2} \tag{14}$$

when $k_y \leqslant 0$ (See Fig. 6.) Then we get

$$dk_y = k(k^2 - k_x^2)^{-1/2} dk$$

from Eq. (13) and

$$dk_y = -k(k^2 - k_x^2)^{-1/2} dk$$

from Eq. (14). These results allow us to write Eq. (12) as

$$S(k_x) = \int_{k_x}^{\infty} \frac{kS(k,\theta)\,dk}{\sqrt{k^2 - k_x^2}} + \int_{k_x}^{\infty} \frac{kS(k,-\theta)\,dk}{\sqrt{k^2 - k_x^2}} \tag{15}$$

for $k_x \geqslant 0$. For $k_x \leqslant 0$, we can use the symmetry property that $S(-k_x) = S(k_x)$. The presence of $S(k, -\theta)$ in the second integral arises because for integration over the line in the fourth quadrant (see Fig. 6) we have the relation that the polar angle as a function of k must be the negative of the polar angle that appears in the first integrand.

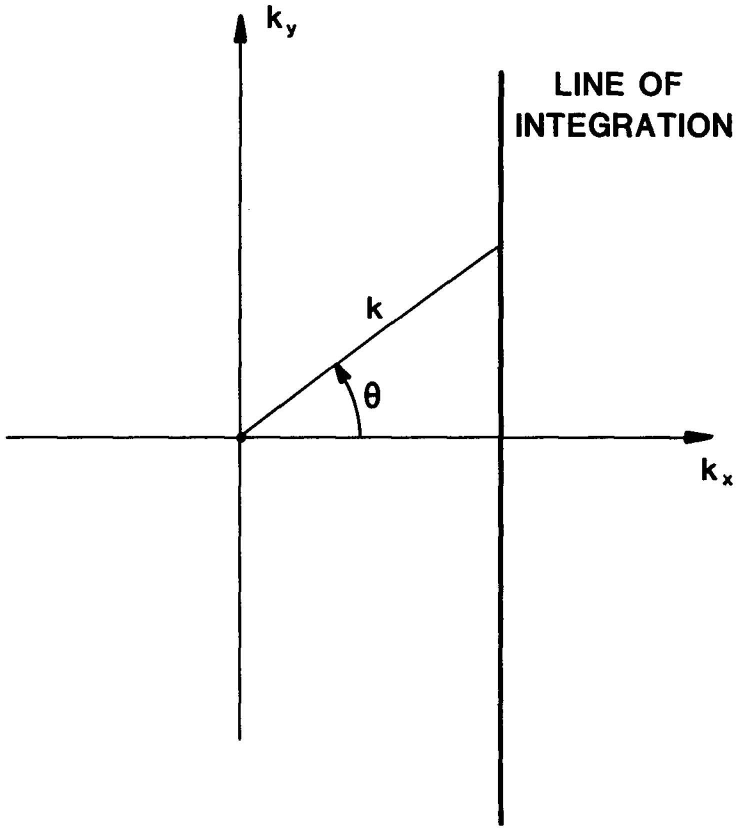

Fig. 6 — Line in the spectral plane along which integration must be performed to obtain the one-dimensional spectrum $S(k_x)$ from the two-dimensional spectrum $S(k_x, k_y)$. The integration is most conveniently done by considering the upper and lower line segments separately. For the upper segment $\theta = \theta(k, k_x)$ while on the lower segment we can write $\theta = -\theta(k, k_x)$. On each line segment k will vary from k_x to ∞.

It is now useful to obtain a form of the result that applies when the function $S(k,\theta)$ does not depend on the angle θ, so that we can write $S(k,\theta) = S(k)$. This is the special case when contours of the two-dimensional spectrum $S(k_x,k_y)$ are circles in the k_x,k_y plane and arises in many situations of practical interest. We can refer to this as the isotropic case and thereby define isotropy in terms of spectral behavior. A more standard procedure is to define isotropy in terms of the correlation function of the process, but that route would take us too far off the path of the present discussion.

We get from Eq. (15) for the case of independence of angle the result that

$$S(k_x) = 2 \int_{|k_x|}^{\infty} \frac{kS(k)\,dk}{\sqrt{k^2 - k_x^2}}, \tag{16}$$

where the absolute value allows the expression to be used for positive or negative values of k_x. The quantity k_x enters the integrand as a fixed parameter and is also a limit on the integral. Equation (15) specifies the relationship between the one-dimensional spectrum $S(k_x)$ and the two-dimensional spectrum $S(k)$, which is the two-dimensional spectrum in the k_x,k_y plane written as a function of k.

We can also establish a relationship between $S(k_x)$ and the isotropic two-dimensional spectrum $S(k)$. From Eq. (8) we have the simple relation that

$$2\pi\, kS(k) = S(k). \tag{17}$$

This enables Eq. (16) to be written as

$$S(k_x) = \frac{1}{\pi} \int_{|k_x|}^{\infty} \frac{S(k)\,dk}{\sqrt{k^2 - k_x^2}}, \tag{18}$$

which gives an important relation between the one-dimensional spectrum and the isotropic spectrum $S(k)$ that refers to the complete process, occurring over two dimensions.

Equation (18) gives rise to the question as to whether the relationship can be inverted so as to obtain $S(k)$ from a known $S(k_x)$. This problem arises in practice when the process is known, or assumed, to be isotropic but only one-dimensional data are available to describe the process. The inversion is made possible by the theory associated with Abel's integral equation, of which Eq. (18) is a particular example [8,9]. The theory yields the inverse result that

$$S(k) = -2 \int_{k}^{\infty} \frac{k}{\sqrt{k_x^2 - k^2}} \frac{dS(k_x)}{dk_x}\, dk_x. \tag{19}$$

Thus for an isotropic process, the spectra $S(k)$ and $S(k_x)$ can be regarded as an Abel transform pair. For some examples of the formal use of the Abel transform relation in various fields of geophysics [10,11].

It is desirable to give some rationale for why there should be such an inverse relationship. The appeal to Abel's integral equation is essentially a formal mathematical treatment of the issue which offers no intuitive basis for understanding the existence of the relationship. Figure 7 shows a useful way to think of the spectral plane for establishing an intuitive understanding of the inverse relations. The plane is divided into circular bands of thickness dk which have a mean energy content of $S(k)dk$ and a set of vertical bands of thickness dk_x (where $dk_x = dk$) with a mean energy content of $S(k_x)dk_x$. It is clear from the figure that the circular bands establish a partition of the vertical bands. Further, the mean energy of a vertical band can be calculated by summing the contributions from the various regions of the circular bands which make up the partition. This will obviously involve geometric factors and values of $S(k)$ for $k \geqslant k_x$.

The following shows how the inverse result of expressing $S(k)$ in terms of $S(k_x)$ seem reasonable (see Fig. 7). Consider the set of values $S(k_x)$ where k_x ranges from k to ∞ in increments dk. Each value $S(k_x)$ can be represented as a sum over $S(k)$ values, as previously discussed. Thus there is a matrix relation between the infinite set of $S(k_x)$ values and the infinite set of $S(k)$ values. To represent $S(k)$ in terms of $S(k_x)$ it is necessary to invert the matrix relationship. The result of inverting the matrix must take the form of Eq. (19) when transition is made from a summation to an integral. These remarks are intended to indicate a physical justification for the inverse result. They do not constitute a mathematical derivation of the result but do suggest the lines along which such a derivation may be pursued.

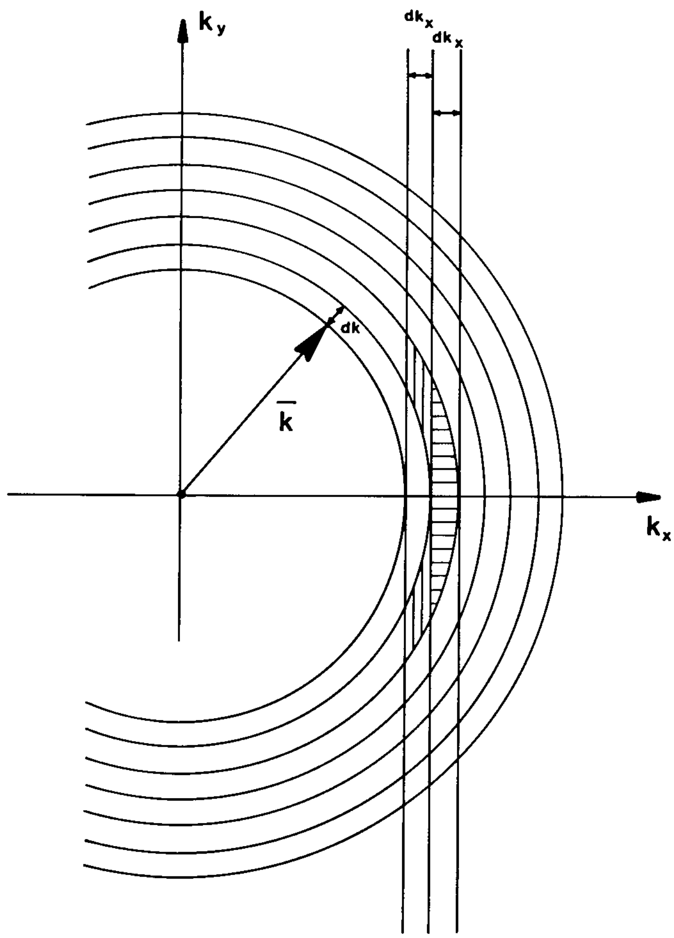

Fig. 7 — Illustration to give rationale for expecting an inverse relationship to exist between $S(k_x)$ and $S(k)$. The vertical bands contain an energy $S(k_x)dk_x$ while the circular bands contain an energy $S(k)dk$, where $dk = dk_x$ for the situation shown. The circular bands establish a partition of the vertical bands which makes it clear that $S(k_x)dk_x$ is the sum of contributions arising from the circular bands. The portion of each circular band with must be taken is determined from geometric considerations.

Note Eqs. (18) and (19) are fully equivalent to the forms obtained by Bell [10] starting with the correlation function. To make our equations conform with the results of Ref. 10, it is necessary to convert the spectrum $S(k_x)$ to a one-sided form by placing all the spectral energy into positive values of the wavenumber k_x. This would introduce a factor of two and make our results identical with Bell's.

It is useful to summarize the relations that are known to exist between the different spectral quantities. This is done in Table 1 for the general case and in Table 2 for the special case of isotropy. An important aspect of Table 1 is that some of the relations are not known in general because of the nonexistence of appropriate inverse results. Table 2, on the other hand, has explicit formulas for all the entries, this is entirely a consequence of the assumption of isotropy. The situation becomes more involved with processes that depend on more than two independent variables, since spectra could be defined to account for several different directional aspects of the spectral energy. In such cases it is helpful to keep in mind the matrix of possible relations corresponding to Tables 1 and 2 for the two-dimensional case.

Table 1 — Relations between the spectral quantities for the general case. The table should be read as the quantity appearing at the left of a row and column entry is equal to the specified function of the quantity that appears at the top of the column. Note that a column is the set of inverse relations for entries appearing in the row having the same label as the column. Some entries indicate no general relation because an inverse relation is needed for which no general result is known.

	$S(k_x,k_y)$ $S(k,\theta)$	$\mathcal{S}(k,\theta)$	$\mathcal{S}(k)$	$S(k_x)$
$S(k_x,k_y)$ $S(k,\theta)$	$\equiv$	$\frac{1}{k}\mathcal{S}(k,\theta)$	NO GENERAL RELATION	NO GENERAL RELATION
$\mathcal{S}(k,\theta)$	$kS(k,\theta)$	$\equiv$	NO GENERAL RELATION	NO GENERAL RELATION
$\mathcal{S}(k)$	$\int_0^{2\pi} S(k,\theta)k\,d\theta$	$\int_0^{2\pi} \mathcal{S}(k,\theta)\,d\theta$	$\equiv$	NO GENERAL RELATION
$S(k_x)$	$\int_{-\infty}^{\infty} S(k_x,k_y)\,dk_y$	$\int_{\lvert k_x\rvert}^{\infty} \frac{\mathcal{S}(k,\theta)\,dk}{\sqrt{k^2-k_x^2}} + \int_{\lvert k_x\rvert}^{\infty} \frac{\mathcal{S}(k,-\theta)\,dk}{\sqrt{k^2-k_x^2}}$	NO GENERAL RELATION	$\equiv$

Table 2 — Relations between the spectral quantities for the special case of isotropy, for which $S(k_x,k_y)$ may be written entirely as a function of k. This function is designated as $S(k)$, which represents a change in notation from Table 1. The other quantities are designated as in Table 1. The table should be read as the quantity appearing at the left of a row and column entry is equal to the specified function of the quantity that appears at the top of the column. Note that a column is the set of inverse relations for entries appearing in the row having the same label as the column and that all the inverse relations are known for the isotropic case.

	$S(k_x,k_y)$ $S(k)$	$\mathcal{S}(k,\theta)$	$\mathcal{S}(k)$	$S(k_x)$
$S(k_x,k_y)$ $S(k)$	$\equiv$	$\frac{1}{k}\mathcal{S}(k,\theta)$	$\frac{1}{2\pi k}\mathcal{S}(k)$	$-\frac{1}{\pi}\int_k^\infty \frac{1}{\sqrt{k_x^2-k^2}}\frac{dS(k_x)}{dk_x}dk_x$
$\mathcal{S}(k,\theta)$	$kS(k)$	$\equiv$	$\frac{1}{2\pi}\mathcal{S}(k)$	$-\frac{1}{\pi}\int_k^\infty \frac{k}{\sqrt{k_x^2-k^2}}\frac{dS(k_x)}{dk_x}dk_x$
$\mathcal{S}(k)$	$2\pi k\ S(k)$	$2\pi\mathcal{S}(k,\theta)$	$\equiv$	$-2\int_k^\infty \frac{k}{\sqrt{k_x^2-k^2}}\frac{dS(k_x)}{dk_x}dk_x$
$S(k_x)$	$\int_{-\infty}^{\infty} S(k_x,k_y)dk_y$	$2\int_{\lvert k_x\rvert}^\infty \frac{\mathcal{S}(k,\theta)dk}{\sqrt{k^2-k_x^2}}$	$\frac{1}{\pi}\int_{\lvert k_x\rvert}^\infty \frac{\mathcal{S}(k)dk}{\sqrt{k^2-k_x^2}}$	$\equiv$

EVALUATION OF THE SPECTRA FOR POWER-LAW MODELS

An important and useful analytical representation of spectra is offered by power-law approximations to empirical spectra. This is especially true for physical processes that are of a wide-band nature and for which the spectral energy is distributed over the band with no strong preference for isolated wavenumbers.

In particular, for the two-dimensional wavenumber spectrum $S(k_x,\ k_y)$ it is sometimes useful to use the representation

$$S(k_x,\ k_y) = S_0 k^{-\alpha}, \tag{20}$$

where one might in general assume that S_0 and α are functions of the angle θ in the spectral plane. This type of representation would be useful in those situations where a strict power-law behavior may be applicable along individual rays from the origin but with the parameters of the power law depending on which particular ray is being used.

Little additional simplification can be achieved with the expression in Eq. (20) when S_0 and α are allowed to vary with θ. Now consider the special case of isotropy for which the two-dimensional spectrum $S(k_x, k_y)$ can be represented as in Eq. (20) with constant values of S_0 and α.

The scalar wavenumber spectrum is given by Eq. (8) as

$$S(k) = 2\pi kS,$$

or

$$S(k) = 2\pi S_0 k^{-\alpha+1}. \tag{21}$$

An important aspect of the scalar wavenumber spectrum $S(k)$ is that the exponent in the power law is different from the exponent which appears in the expression for $S(k_x, k_y)$.

Another spectrum of interest is the one-dimensional spectrum $S(k_x)$ that would arise in a study of behavior of the process $\eta(x, y)$ along the x-axis. Since the process is isotropic, there is no special significance attached to the x-axis; the statistical behavior of the process along any line in space will be identical with the behavior along any other line in space. There are several ways to obtain $S(k_x)$, but it is perhaps simplest to derive it from the basic relation

$$S(k_x) = \int_{-\infty}^{\infty} S(k_x, k_y)\, dk_y.$$

Using the form in Eq. (20) and the symmetry in k_y, this becomes

$$S(k_x) = 2\int_0^{\infty} S_0 k^{-\alpha} dk_y. \tag{22}$$

The evaluation of this integral is carried out by considering k_x as positive and using the substitutions

$$k = \sqrt{k_x^2 + k_y^2},$$

$$k_y = k_x \tan\theta,$$

$$dk_y = k_x \sec^2\theta\, d\theta.$$

The result is that Eq. (22) becomes

$$S(k_x) = 2S_0|k_x|^{-\alpha+1} \int_0^{\frac{\pi}{2}} \cos^{\alpha-2}\theta\, d\theta. \tag{23}$$

The one-dimensional spectrum is written in a form applicable to positive or negative k_x. It has a power-law behavior that differs from that of the two-dimensional spectrum by having an exponent that is increased by 1. The situation is quite similar to the relationship between the two-dimensional spectrum and the scalar wavenumber spectrum $S(k)$. The only difference between the mathematical expressions for $S(k_x)$ and $S(k)$ involves a constant factor that arises because these two spectra refer to different types of energy associated with the process. This difference is indicated in the section on various forms of spectral energy.

It is desirable to evaluate the one-dimensional spectrum explicitly for any situation that might arise in applications. To do this we write Eq. (23) as

$$S(k_x) = 2S_0|k_x|^{-\alpha+1} f(\alpha - 2), \tag{24}$$

where

$$f(\alpha) = \int_0^{\pi/2} \cos^{\alpha}\theta\, d\theta. \tag{25}$$

Now consider properties of the function $f(\alpha)$ and the evaluation of the function for specific values of α.

The first question to ask is the range of values for which the integral in Eq. (25) makes sense. For values $\alpha \leqslant -1$ the integrand in Eq. (25) becomes infinite at $\frac{\pi}{2}$ in such a way that $f(\alpha)$ would be

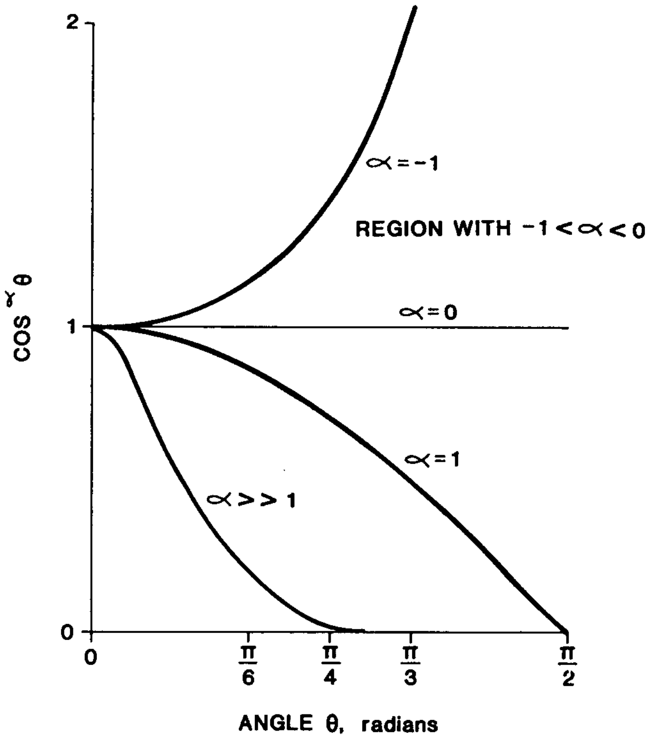

Fig. 8 — Behavior of the function $\cos^\alpha\theta$ for various values of α

infinite. Consequently, it is only meaningful to consider values of $\alpha > -1$. Figure 8 shows an indication of the behavior of the integrand in Eq. (25).

One important general property of the function $f(\alpha)$ is that it is a decreasing function of α. This is seen by evaluating the derivative of the function. We get

$$\frac{d}{d\alpha}f(\alpha) = \frac{d}{d\alpha}\int_0^{\pi/2} \cos^\alpha\theta d\theta$$

$$= \int_0^{\pi/2} \cos^\alpha\theta \log \cos \theta \, d\theta. \tag{26}$$

The integrand in Eq. (26) is negative except possibly at an endpoint of the integration interval. Consequently, we have

$$\frac{d}{d\alpha}f(\alpha) < 0,$$

which establishes the decreasing nature of the function in the interval $\alpha > -1$.

For integer values, the function $f(\alpha)$ can be evaluated directly by a repeated application of the technique of integration by parts. This yields the result that

$$f(2n) = \int_0^{\pi/2} \cos^{2n} \theta d\theta$$

$$= \frac{2n-1}{2n}\int_0^{\pi/2} \cos^{2n-2}\theta d\theta$$

$$= \frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{1}{2}\frac{\pi}{2}, \tag{27}$$

where there are n factors ($n \geqslant 0$) preceding the last factor of $\pi/2$. The result for an odd value of α is derived similarly and we get

$$\begin{aligned} f(2n+1) &= \int_0^{\pi/2} \cos^{2n+1} \theta d\theta \\ &= \frac{2n}{2n+1} \int_0^{\pi/2} \cos^{2n-1} \theta d\theta \\ &= \frac{2n}{2n+1} \frac{2n-2}{2n-1} \cdots \frac{2}{3} \end{aligned} \tag{28}$$

where n factors ($n \geqslant 0$) appear on the right-hand side of Eq. (28).

Some general relations can be established between $f(\alpha)$, the beta function, and the gamma function which can be used for evaluating $f(\alpha)$ at noninteger values of the argument. An integral representation of the beta function is

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1} \theta \cos^{2y-1} \theta \, d\theta \tag{29}$$

where x,y are positive. The beta function is related to the gamma function by

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{30}$$

where the gamma function satisfies

$$\Gamma(x+1) = x\,\Gamma(x). \tag{31}$$

A comparison of the integral expression for $f(\alpha)$ with the integral representation for the beta function in Eq. (29) leads to the result that

$$f(\alpha) = \frac{1}{2} B \left(\frac{1}{2}, \frac{\alpha+1}{2} \right), \tag{32}$$

which can be expressed in terms of the gamma function as

$$f(\alpha) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)}. \tag{33}$$

The notation and properties of the beta and gamma functions are given in Ref. 12.

Equation (33) provides the most useful means for evaluating $f(\alpha)$ for general values of α. One can simply use Eq. (31) in conjunction with tabulated values of the gamma function for some standardized interval. Figure 9 is a plot of the function $f(\alpha)$ over a limited range of α. It agrees with properties already derived, from the above results. The function $f(\alpha)$ is: (a) positive-valued, (b) tends to infinity at $\alpha = -1$, (c) steadily decreases in value as α increases, (d) changes slowly in value when $\alpha >> 1$, and (e) asymptotically approaches zero as α approaches infinity.

COMMENTS ON OCEAN SURFACE WIND-WAVE SPECTRA

An interesting and instructive example of the use of spectra for a process that depends on several variables arises in the study of ocean surface waves [1,2]. A strong motivation for consideration of this case comes from the fact that ocean surface waves are a common aspect of our experience with at least some aspects of the process readily observable. We shall use ocean surface waves as a specific geophysical model to illustrate some important general points concerning spectral analysis for random processes dependent on several variables.

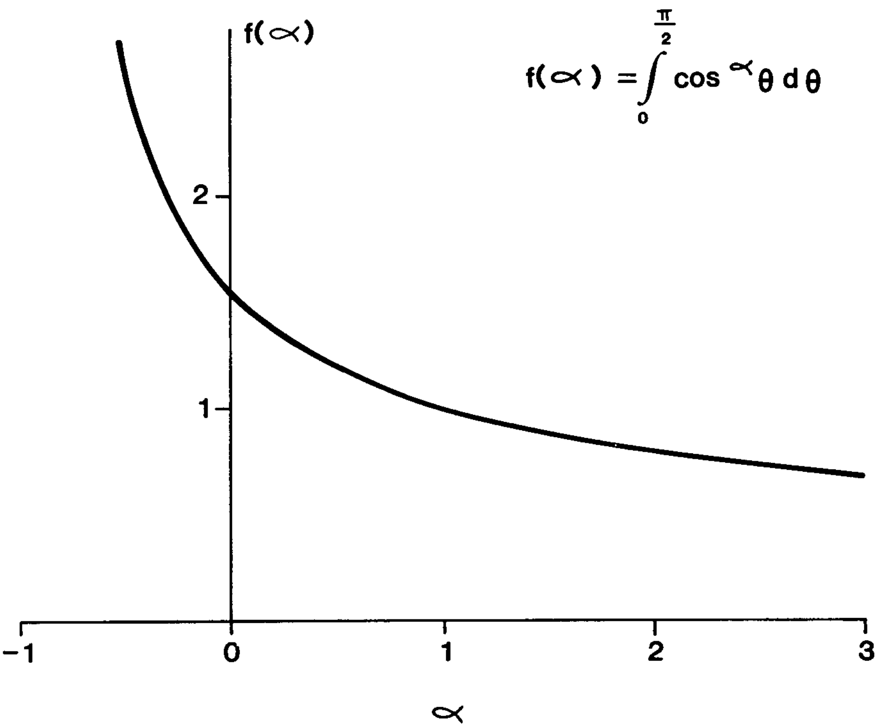

Fig. 9 — The behavior of the function $f(\alpha)$ which is related to the one-dimensional spectrum through $S(k_x) = 2S_0|k_x|^{-\alpha+1}f(\alpha - 2)$

The theoretical investigation of the spectral characteristics of ocean surface waves uses a representation of the elevation of the ocean surface above some mean level given by a Fourier-Stieltjes integral,

$$\eta\ (\mathbf{x},t) = \int_{\mathbf{k}}\int_{m} dA\,(\mathbf{k},m)\ \exp\left\{i(\mathbf{k}\cdot\mathbf{x} - mt)\right\}, \tag{34}$$

where the integration is over all wavenumber, frequency space. This representation is obtained on the basis that the process may be considered as a stationary random function of both position and time. An important consequence of the stationarity is that the Fourier-Stieltjes coefficients satisfy

$$\begin{aligned} \overline{dA\,(\mathbf{k},m)\,dA\,(\mathbf{k}',m')} &= 0 \text{ if } \mathbf{k},m \neq \mathbf{k},m' \\ &= X(\mathbf{k},m)\,dkdm \text{ if } \mathbf{k} = k',m = m'. \end{aligned} \tag{35}$$

It can be shown that

$$\overline{\eta^2} = \int_{\mathbf{k}}\int_{m} X(\mathbf{k},m)\,d\mathbf{k}dm \tag{36}$$

so that $X(\mathbf{k},m)$ is the wavenumber-frequency spectrum.

If it is assumed that the ocean wave system consists of individual wave components whose amplitudes are sufficiently small, then in deep water we have the dispersion relation

$$\sigma^2 = gk \tag{37}$$

between the frequency σ of an individual component and the magnitude of its wavenumber. A consequence of the dispersion relation is that spectral energy in wavenumber-frequency space can only exist in certain regions. Specifically, we must have

$$X(\mathbf{k},\ m) = \psi(\mathbf{k})\delta(m - \sigma), \tag{38}$$

where σ is the function of k given in Eq. (37). When this result is substituted in Eq. (36), the result is that

$$\overline{\eta^2} = \int_{\mathbf{k}} \psi(\mathbf{k})\, d\mathbf{k}, \tag{39}$$

where $\psi(\mathbf{k})$ is the wavenumber spectrum.

Our main reason for considering the ocean wave case is that it demonstrates some interesting aspects of spectral analysis. For example, suppose that one made detailed measurements of the ocean surface elevation at one instant of time over a sufficiently large area of the surface. Such data would allow an estimate of the wavenumber spectrum $\psi(\mathbf{k})$, where the result would be obtained that

$$\psi(\mathbf{k}) = \psi(-\mathbf{k}). \tag{40}$$

That is, the spectral energy appearing at **k** is the same as at $-\mathbf{k}$ (see Fig. 10). An actual example of the wavenumber spectrum derived from field data is shown in Ref. 1. The reason for the equal distribution of spectral energy at **k** and $-\mathbf{k}$ is due to the limited information assumed available for the process. An instantaneous snapshot of the sea surface does not allow one to distinguish between waves traveling in the $+\mathbf{k}$ and $-\mathbf{k}$ directions. Additional information about the process would be needed to fully describe the propagational characteristics of the wave energy. This exemplifies the important point that the particular spectrum, or spectral characteristics, one can study depends on the nature of the measurements.

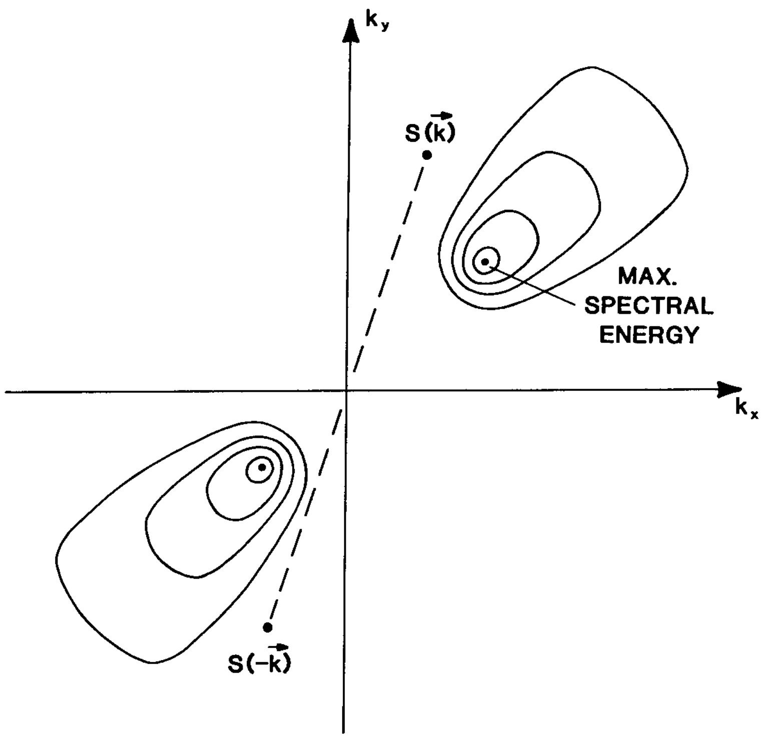

Fig. 10 — Schematic illustration of the anisotropic nature of the wavenumber spectrum $S(\vec{k})$ for ocean surface wind-waves. The indicated spectrum is representative of what would be calculated from a "snapshot" of the ocean surface. The spectrum has a symmetry $S(-\vec{k}) = S(\vec{k})$ as a result of the type of data assumed, so any half-plane will be an image of the other half-plane.

Perhaps the most significant aspect of the spectrum is its obvious lack of isotropy. The variation of the spectral energy density along a ray from the origin will be found to change with direction, or the particular ray being examined. Another aspect of such spectra is that the spectrum does not continually increase as the wavenumber is reduced. This particular pattern of behavior is unusual for geophysical processes, which often demonstrate the characteristics of a red-noise process.

Theoretical spectra used for the description of ocean surface waves are often constrained in such a way as to require that spectral components be traveling in the downwind direction. This particular constraint arises from observational studies of surface waves as well as from energetic considerations that identify the wind as the primary source of energy for the surface waves. The restriction can be put in the form that nonvanishing spectral energy should only appear at wavenumbers with

$$\mathbf{k} \cdot \mathbf{U} \geqslant 0,$$

where $\mathbf{U}$ is the wind vector at some canonical height above the ocean surface. This allows the wavenumber spectrum to be represented as in Fig. 11, where there is now no ambiguity in the spectral plane. Theoretical surface wave spectra are usually represented so that a symmetry in the spectrum occurs relative to the wind direction, which is also indicated in the figure. An immediate implication of the figure is that the natural coordinate system to employ in studies of the two-dimensional statistical structure of ocean surface waves would have one axis aligned with the wind direction. The existence of a preferred direction oriented with the wind direction also occurs in more general spectral wave models [13].

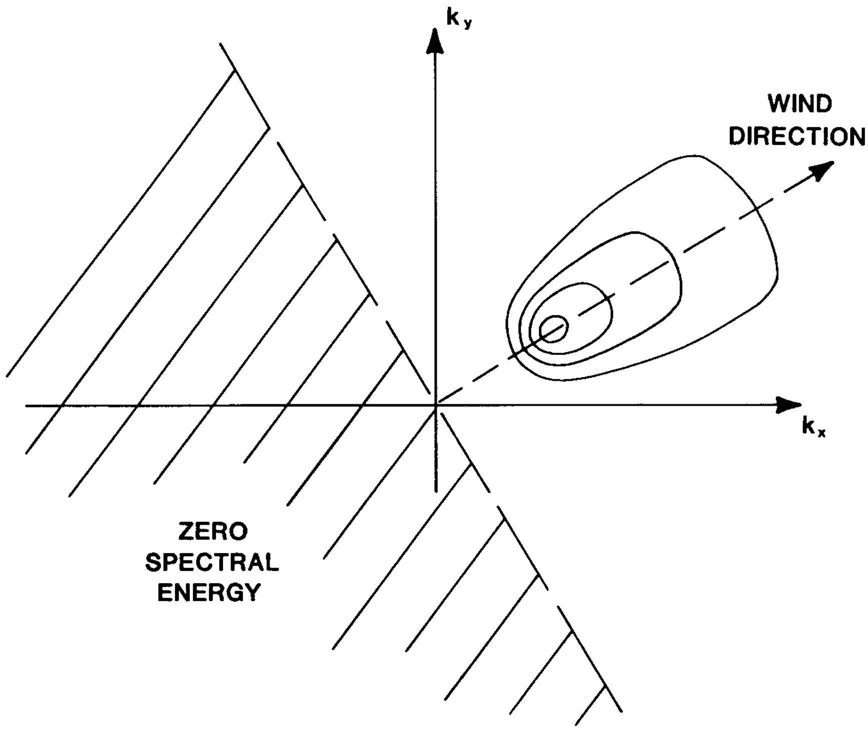

Fig. 11 — Schematic illustration of the wind-wave spectrum under the assumption of down-wind propagating waves

The importance of these aspects of ocean surface waves is that they exemplify certain features of random processes that vary over a multidimensional space that must be dealt with in other situations not as well understood. Consider the possible existence of an orientation direction for such processes. Orientation does not arise as an issue in truly one-dimensional processes. We here argue that anisotropy must be dealt with on an equal footing with stationarity in the treatment of data. The reason for this is that in the collection of several separate samples to form an ensemble, the data samples must be oriented in the same way relative to one another for ensemble averages to properly represent statistical structure associated with the process.

An indication of the problem can be obtained by thinking of the surface wave case. Suppose one collected finite two-dimensional samples of the ocean surface wave elevation from many areas of the world's oceans. These samples could be used to construct an ensemble as indicated in Fig. 12. Suppose that only samples with the same fetch, duration, and wind speed were placed in the ensemble with the additional restriction that all samples would pertain to deep water. We are artificially creating an ensemble in which only the wind direction has not been properly considered. The ensemble average spectrum for the ensemble indicated in the figure would turn out to be in error. In effect, the correct spectrum indicated in Fig. 11 would be spread over the spectral plane and much reduced in amplitude. This effect would entirely be the result of the failure to align the data samples with the same orientation.

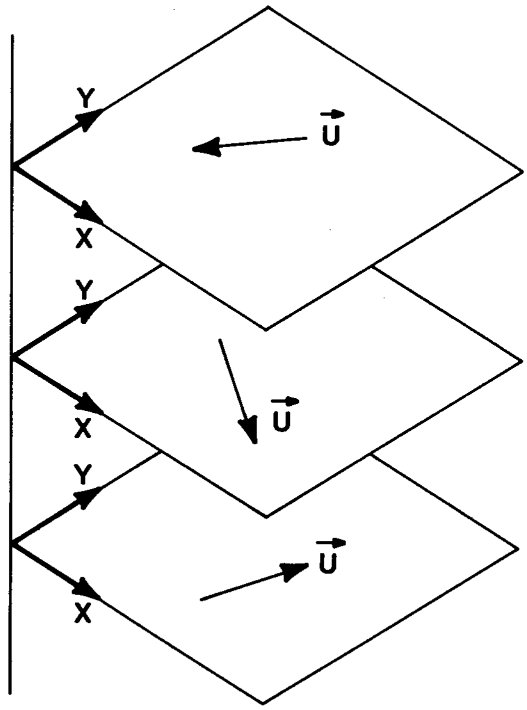

Fig. 12 — Schematic illustration of the nature of ensemble in which direction was not taken account of properly

Although the above example is rather artificially contrived, it does dramatize the serious problems associated with orientation. For a process dependent on several variables and for which little is known concerning orientation, it is clear that techniques are needed for determining orientation properties of the process. There is also a practical necessity for being able to determine orientation in the earliest stages of the data processing if an ensemble type approach is to be used. This problem does not seem to have been adequately addressed in the literature on stochastic processes.

There are, then, certain general aspects of ocean-surface-wave spectra that are useful in a conceptual way for the study of other processes. The main features of the ocean-surface wavenumber spectrum are the existence of a preferred direction and the location of the maximum in the spectral energy density at a wavenumber whose magnitude changes with wind conditions. The description of ocean surface waves as a stochastic process is in a fairly advanced state relative to the description of other geophysical processes and represents an example of a process dependent on several dimensions which is useful for dramatizing certain general issues associated with such processes.

CONCLUSIONS

We have considered a variety of spectra which arise in the discussion of multidimensional stochastic processes with particular emphasis on the two-dimensional situation. The number of spectra that can be used in the description and analysis of such processes is large and can be a source of confusion. We have stressed the need to keep in mind the physical interpretation of the different spectra, which are related to different ways of describing the spectral energy. The basic reason for the large number of possible spectral quantities for such multidimensional processes is that it is possible to view the process in different ways. A point worth stressing is the necessity to keep the full nature of the process in mind.

The various spectra that are introduced are all related to the basic spectrum $S(k_x, k_y)$ of the process $\eta(x, y)$. A particular question of interest is establishing inverse relationships where possible. An inverse relationship of particular importance is that between the one-dimensional spectrum and the two-dimensional isotropic spectrum in the case of an isotropic process for which the two-dimensional spectrum is independent of angle in the spectral plane. This is done through appeal to Abel's integral equation, which indicates that $S(k_x)$ and $S(k)$ may be thought of as an Abel transform pair. A physical justification is given for the existence of the inverse relation to provide an intuitive understanding of how the Abel transform relation arises. The physical justification is in terms of an examination of two ways of describing the distribution of spectral energy in the spectral plane. It represents a somewhat novel way of thinking of the inverse relation and seems capable of being put on a firm mathematical basis.

We consider the important case of power-law models for the two-dimensional spectrum because of its expected use in representing two-dimensional spectral behavior. The power-law representation is mainly considered for isotropic processes. Again, the one-dimensional spectrum requires some involved calculations. The one-dimensional spectrum is related to a definite integral containing the exponent of the power-law as a parameter. Some general properties are established for the integral. An explicit evaluation is given in terms of the gamma function. The definite integral is also plotted as a function of the parameter to provide a useful means for obtaining values in particular cases which might arise in applications.

The representation of ocean surface wind-waves as a stationary process is considered because the wind-wave case offers a useful example of a process taking place over a multidimensional space. Its utility stems from the rather extensive work done on the problem and because of the ease of observing some aspects of the process, even during routine daily life. The windwave case is also important because it serves as an example of strong anisotropy of the two-dimensional spectrum relative to the direction of the wind. A general implication of the discussion of wind-waves is that anisotropy of the process must be considered equally important with stationarity in the treatment of data, especially if ensemble averages are to be used to estimate the two-dimensional spectrum. This is due to the need to properly orient the different realizations that make up the ensemble prior to averaging. A failure to properly account for orientation can lead to ensemble average spectra that grossly misrepresent the statistical structure of the process.

REFERENCES

1. B. Kinsman, *Wind Waves* (Prentice-Hall, Englewood Cliffs, New Jersey, 1965).
2. O.M. Phillips, *The Dynamics of the Upper Ocean*, 2nd ed. (Cambridge University Press, Cambridge, 1977).
3. National Academy of Sciences, *Ocean Wave Spectra (Prentice-Hall, Englewood Cliffs, New Jersey, 1963).*
4. A.M. Yaglom, *An Introduction to the Theory of Stationary*
5. Random Functions (Dover Publications, Inc., New York, 1973).
6. G.K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, 1960).
7. J.E. Moyal, "Stochastic Processes and Statistical Physics," *J. R. Statist. Soc. B* **11**, 150-210 (1949).
8. J.L. Lumley, *Stochastic Tools in Turbulence* (Academic Press, New York, 1970).
9. I.N. Sneddon, *The Use of Integral Transforms* (McGraw-Hill, New York, 1972).

10. F.G. Tricomi, *Integral Equations* (Interscience, New York, 1957).

11. T.H. Bell, Jr., "Statistical Features of Sea-Floor Topography," *Deep-Sea Res.* **22**, 883-892 (1975).

12. R.V. Sailor, "Relationships Between Along-Track and Two-Dimensional Spectra of Geophysical Data: Application s in Geodesy," *EOS, Trans. Am. Geophys. Union* **61**, 937 (1980).

13. M. Abramowitz, and I.A. Segun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1965)

14. G.L. Tyler, C.C. Teague, R.H. Stewart, A.M. Peterson, W.H. Munk, and J.W. Joy, "Wave Directional Spectra from Synthetic Aperture Observations of Radio Scatter," *Deep-Sea Res.* **21**, 989-1016 (1974).

GLOSSARY

$dA(\mathbf{k}), dA(\mathbf{k},m)$	Fourier-Stieltjes coefficients
$B(x,y)$	beta function
$f(\alpha)$	special function entering spectral calculation
g	acceleration of gravity
$\mathbf{k}$	vector wavenumber
k_x, k_y	components of wavenumber
m	frequency
S_0	parameter entering power law
$S(\mathbf{k}), S(k_x,k_y)$	two-dimensional wavenumber spectrum
$S(k,\theta)$	two-dimensional wavenumber spectrum expressed in polar coordinates
$S(k,\theta)$	directional spectrum
$S(k)$	scalar wavenumber spectrum
$S(k_x)$	one-dimensional spectrum
$\mathbf{U}$	wind vector
$\mathbf{x}$	position vector
x,y	spatial coordinates
$X(\mathbf{k},m)$	wavenumber-frequency spectrum for wind-waves
α	exponent of power-law
$\Gamma(x)$	gamma function
$\delta(x)$	Dirac delta function
η	the fundamental variable of the stochastic process of interest, taken to be real-valued
$\overline{\eta^2}$	mean square of η, sometimes more casually referred to as the mean energy
θ	angle in spectral plane
σ	frequency
ϕ	phase
$\psi(\mathbf{k})$	wavenumber spectrum for wind-waves
$\overline{()}$	average of ( )
$(\ )^*$	complex conjugate of ( )